

Statistical relativistic temperature transformation for ideal gas of bradyons, luxons and tachyons

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Abstract. - Starting from a microcanonical statistical approach and special relativity, the relativistic transformations for temperature and pressure for an ideal gas of bradyons, luxons or tachyons is found. These transformations are in agreement with the three laws of thermodynamic and our temperature transformation is the same as Ott's. Besides, it is shown that the thermodynamic dS element is Lorentz-invariant.

Introduction. – The relativistic transformation of temperature is a problem which has been controversial for almost a century. There has been many proposals, starting by pure classical thermodynamics [1–6] until classical and quantum statistical mechanics [7–11]. Starting from different postulates, each one of this works had tried to establish how the different thermodynamics quantities change under Lorentz transformations, but they have obtained incompatible results. For example, in Refs. [12, 13] a review of different formalisms is done. In particular, in Ref. [12] it is established that the different formalisms are mathematically equivalent to each other, because there is a one to one correspondence between the quantities defined in every formalism.

The idea to generalize the statistical mechanics to relativistic systems dates from Jüttner [15, 16], who proposed a relativistic form of Maxwell-Boltzmann velocities distribution. Other attempts to get the correct relativistic distribution function that fits correctly experimental data have been recently done. For example, in Refs. [9, 10] a new mathematical formalism was created in order to develop a non-extensive relativistic statistical mechanics under a canonical ensemble, which works to fit data of cosmic rays. Recently, it has been shown through numerical simulation that Jüttner's distribution function is the distribution in special relativity that produces the best fit for a dilute gas of two components mixture with collisions in one dimension [11].

In other hand, other works conduct to others distri-

bution functions than Jüttner one. For instance, in Ref. [14], authors perform numerical simulations of electrons accelerated to relativistic energies due to its interaction with waves generated by longitudinal streaming plasma instabilities. They found an equilibrium distribution which present power-law tails at high energies. Although Refs. [11, 14] consider different systems, both show that the old problems of transformation of temperature and pressure, and the form of distribution function in theoretical relativistic statistical mechanics appears in numerical simulation.

However, in many of these works the temperature transformation are assumed and not derived from the theory itself. This happen because in canonical ensembles the temperature $\beta = T^{-1}$ is a system variable (we set Boltzmann constant $k_B = 1$). Therefore, it is not easy to find a temperature transformation between two inertial frames moving with relative velocities by direct calculation.

To overcome this problem, in this article we calculate the temperature in the microcanonical ensemble of a relativistic ideal gas of bradyons, luxons or tachyons. In this ensemble the intensive quantities are not variables and it is possible to find the temperature only by taking derivatives. Thus, the calculations are simpler than in a canonical ensemble because we only need to fix the energy of these particles. In addition, according to Gibbs' postulate, the results should be independent from the ensemble used to calculate it. This postulate allow us to obtain a result that is equivalent to the one obtained in any other

ensemble [17].

We are extending the old problem of how the temperature of bradyons transform in different frames to luxons and tachyons. The reason to include tachyons under this study is the wide range of relativistic systems in which they can be included. They play an important role in recent developments in inflationary cosmological models [18–20], string theory black holes models [21, 22] and there are, even, proposed procedures to measure tachyonic states [23].

To find the temperature transformation we first derive the microcanonical entropy of the systems. Then we calculate the temperature in a thermodynamic way showing how it transforms. Furthermore, we show that the entropy thermodynamic dS element is Lorentz invariant for each particle specie.

Entropy calculation. – Consider an ideal gas (of bradyons, luxons or tachyons) which is at rest in a inertial frame I . Let us suppose other inertial frame I' moving with constant velocity $\mathbf{w} = w\hat{x}$ respective to I . Setting $c = 1$, we choose the magnitude $w \leq 1$ if the particles of the systems are bradyons or luxons, and $w > 1$ if the particle system are tachyons.

A bradyon is a particle with rest mass m which moves slower than speed of light. Its dispersion relation is given by

$$p_\mu p^\mu = m^2, \quad (1)$$

where $p_\mu = (\epsilon, \mathbf{p})$ is the 4-momentum of the particle with energy ϵ and momentum \mathbf{p} . We use the signature $(+, -, -, -)$ for our calculations.

A luxon is a particle with null mass which moves at the speed of light. Its dispersion relation has the form

$$p_\mu p^\mu = 0. \quad (2)$$

Finally, a tachyon is a particle with imaginary mass $M = im$ (with m a real quantity) which moves faster than speed of light [24–30]. Its dispersion relation is

$$p_\mu p^\mu = -m^2. \quad (3)$$

We calculate the number of states Ω using the microcanonical ensemble. The three-vector phase-space $d^3\mathbf{q}d^3\mathbf{p}$ is Lorentz invariant for bradyons, luxons and tachyons [29].

We consider an ideal gas of bradyons, luxons or tachyons, consisting of N particles ($N \gg 1$) contained in a volume V . The Hamiltonian of N bradyons is

$$H(p_i) = \sum_{i=1}^N \sqrt{|\mathbf{p}_i|^2 + m^2}, \quad (4)$$

where $|\mathbf{p}_i| = (p_{x,i}^2 + p_{y,i}^2 + p_{z,i}^2)^{1/2}$. The Hamiltonian for N luxons is

$$H(p_i) = \sum_{i=1}^N |\mathbf{p}_i|, \quad (5)$$

and the Hamiltonian for N tachyons is

$$H(p_i) = \sum_{i=1}^N \sqrt{|\mathbf{p}_i|^2 - m^2}. \quad (6)$$

Setting $h = 1$, the microcanonical number of states for each specie is given by

$$\begin{aligned} \Omega &= \frac{1}{N!} \int_{E \leq H(p_i) \leq E + \Delta E} d^3\mathbf{q}_1 \dots d^3\mathbf{q}_N d^3\mathbf{p}_1 \dots d^3\mathbf{p}_N \\ &= \frac{V^N}{N!} \int_{E \leq H(p_i) \leq E + \Delta E} d^3\mathbf{p}_1 \dots d^3\mathbf{p}_N. \end{aligned} \quad (7)$$

For simplicity, we first calculate Σ instead Ω , where

$$\Sigma = \frac{V^N}{N!} \int_{H(p_i) \leq E} d^3\mathbf{p}_1 \dots d^3\mathbf{p}_N. \quad (8)$$

The number of states in a energy interval can be calculated from $\Omega = (\partial\Sigma/\partial E)\Delta E$. Thus, we must write the condition $H(p_i) \leq E$ in a $3N$ -dimensional momentum space. For photons $m = 0$, and then

$$H = \sum_i |\mathbf{p}_i| \leq E \quad (9)$$

Now, we seek for the condition for bradyons. Since no direction in space is preferred, let us start supposing n particles with the same momentum \mathbf{p}_0 , and $N - n$ particles without momentum, with $n \leq N$. In this way, the condition for the Hamiltonian is $H = n(|\mathbf{p}_0|^2 + m^2)^{1/2} + (N - n)m \leq E$. Using this, we can obtain

$$\sum_i |\mathbf{p}_i| = n|\mathbf{p}_0| \leq ((E - (N - n)m)^2 - n^2m^2)^{1/2}$$

However, the factor $(E - (N - n)m)^2 - n^2m^2 = (E - Nm)(E - Nm + 2nm) \leq E^2 - N^2m^2$, and then, it is fulfilled

$$\sum_i |\mathbf{p}_i| \leq (E^2 - N^2m^2)^{1/2} \quad (10)$$

even when $n = N$.

Now, we should study what happen when we have different momenta for each particle. An illustrative example is the next case. If we have $N - 1$ particles with the same momentum and one particle with a different momentum, its sum of the norm of all momenta will be always less than the sum of momenta in Eq. (10) owing to the particles obey the condition $H \leq E$. Following this example, the different cases of momentum of each particle will produced a sum which will be less than Eq. (10). So, the condition Eq. (10) is always valid for bradyons.

Using an analogue argument we can obtain the condition for momentum space for tachyons

$$\sum_i |\mathbf{p}_i| \leq (E^2 + N^2m^2)^{1/2} \quad (11)$$

which is fulfilled always.

All these conditions can be easily written for the momentum components. Thus, the sum will go from 1 to $3N$. Written in that form, they will represent a regular geometric body in $3N$ dimensions, which would be a sphere in the case of classical ideal gas. Then, the problem of calculate the integral Eq. (8) is reduced to find the volume of this regular geometric body. Following the procedure described in Ref. [31], we obtain the number of states for bradyons as

$$\Omega = \frac{V^N}{N!} (2\sqrt{3})^{3N} \frac{(E^2 - N^2 m^2)^{3N/2}}{(3N)!}, \quad (12)$$

the number of states for luxons as

$$\Omega = \frac{V^N}{N!} (2\sqrt{3})^{3N} \frac{E^{3N}}{(3N)!}, \quad (13)$$

and the number of states for tachyons as

$$\Omega = \frac{V^N}{N!} (2\sqrt{3})^{3N} \frac{(E^2 + N^2 m^2)^{3N/2}}{(3N)!}. \quad (14)$$

It is straightforward to obtain the entropy as $S = \ln \Omega$ in a microcanonical ensemble. For bradyons the entropy is

$$S = N \ln \left[\frac{V(E^2 - N^2 m^2)^{3/2}}{27N^4} \right] + 3N \ln [2\sqrt{3}e^{4/3}]. \quad (15)$$

In the same way, the entropy for luxons is

$$S = N \ln \left[\frac{VE^3}{27N^4} \right] + 3N \ln [2\sqrt{3}e^{4/3}], \quad (16)$$

and the entropy for tachyons

$$S = N \ln \left[\frac{V(E^2 + N^2 m^2)^{3/2}}{27N^4} \right] + 3N \ln [2\sqrt{3}e^{4/3}]. \quad (17)$$

Temperature transformation. – To find the relation between the temperature of the system in the I frame and the temperature in the I' frame we need to find how to calculate the number of states in I' . According to Liouville theorem [32] the dimensional phase space $d^3\mathbf{p}'d^3\mathbf{q}' = d^3\mathbf{p}d^3\mathbf{q}$ is Lorentz invariant. Using this, the number of states Ω' in the I' frame can be written using the phase space of I frame

$$\begin{aligned} N!\Omega &= \int_I d^3\mathbf{p}d^3\mathbf{q} \\ &\rightarrow N!\Omega' = \int_{I'} d^3\mathbf{p}'d^3\mathbf{q}' = \int_{I'} d^3\mathbf{p}d^3\mathbf{q}, \end{aligned} \quad (18)$$

where the I' subindex means that now the integration is for all \mathbf{p}'_j that satisfy $\sum_{i=1}^N |\mathbf{p}'_i| \leq (E'^2 - N^2 m'^2)^{1/2}$ for bradyons, $\sum_{i=1}^N |\mathbf{p}'_i| \leq E'$ for luxons and $\sum_{i=1}^N |\mathbf{p}'_i| \leq (E'^2 + N^2 m'^2)^{1/2}$ for tachyons in the I' frame.

Due to Eq. (18), the entropy S' calculated in the I' frame has the same form of the entropy S of Eq. (15),

Eq. (16), and Eq. (17), but changing the energy E by the energy E' , and the volume V by the volume V' .

For bradyons and luxons the energy transforms as $E' = \gamma E$, the momentum transforms as $p' = \gamma p$ and the volume transform as $V = \gamma V'$ since the relative movement is in one dimension. The relativistic factor is $\gamma = (1 - w^2)^{-1/2}$ with $w \leq 1$. For this particles we are considering positive energies.

In the case of tachyons, the energy and momentum transformations are $E' = \zeta E$ and $p' = \zeta p$ respectively, where $\zeta = (w^2 - 1)^{-1/2}$ with $w > 1$ [24, 25]. For simplicity, we consider the positive momentum tachyons. Similarly, the volume transformation for tachyons is $V = \zeta V'$ [26]. Note that if in the I frame the energy, the momentum and the volume of tachyons are real quantities, then in I' frame these quantities are still real.

The above energy, momentum and volume transformations are one of the multiple transformations that can be constructed for a Lorentz invariant tachyon-theory [25–28]. Although the present analysis can be done with other transformations, the election of the above one gives back the usual and simpler energy and momentum relations for tachyons [26]. They ensure that the tachyon three-vector phase-space $d^3\mathbf{q}d^3\mathbf{p}$ is invariant.

In order to obtain the temperature, we calculate the thermodynamical variation of entropy. The variation is $dS = dE/T + (P/T)dV$, where the temperature T and the pressure P are defined by [31]

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_V, \quad \frac{P}{T} = \left(\frac{\partial S}{\partial V} \right)_E. \quad (19)$$

In this way, the calculation of the temperature for bradyons, from Eq. (19), is

$$\frac{1}{T} = \frac{3NE}{E^2 - N^2 m^2}. \quad (20)$$

The temperature for luxons is

$$\frac{1}{T} = \frac{3N}{E}, \quad (21)$$

and the temperature for tachyons is

$$\frac{1}{T} = \frac{3NE}{E^2 + N^2 m^2}. \quad (22)$$

Likewise, we can calculate the pressure for bradyons, luxons and tachyons from Eq. (19). This is

$$\frac{P}{T} = \frac{N}{V}, \quad (23)$$

for the three species. It corresponds to the state equation for an ideal gas.

Calculation of the temperature T' for bradyons, luxons or tachyons in the I' frame can be done using Eq. (18) to evaluate the entropy S' . We can write Eq. (19) for intensive quantities in the I' frame. This allow us to express

Eq. (20) for bradyons, Eq. (21) for luxons, and Eq. (22) for tachyons in I' . Thus, we obtain how the temperature T' from I' frame transforms to temperature T in the I frame for bradyon and luxon ideal gas under the transformations for energy and momentum previously established

$$T' = \gamma T, \quad (24)$$

and for tachyon gas

$$T' = \zeta T. \quad (25)$$

The transformations in Eq. (24) and in Eq. (25) implies that the temperature is not a Lorentz invariant. The temperature transformation for bradyons and luxons (24) is in coincidence with Ott's temperature transformation [4] and other previous works [6,7,33,34], and it is in disagreement with Planck's formalism [1–3,8,10,29]. This means that a moving gas of bradyons or luxons appears hotter. The temperature transformation for tachyons (25) is derived for the first time in the knowledge of the authors.

The difference between our approach and other approaches is the definition of temperature. We emphasize that temperature is defined in a thermodynamic and statistical form by Eq. (19). Thus, the correct definition of Eq. (19) leads naturally to the above correct temperature transformations.

We can do the same analysis for the pressure P' in the I' frame using the transformation for the energy and the momentum. In the same way, according to Liouville theorem and using Eq. (19) and Eq. (23) for I' , we can get the transformation of pressure P' from I' frame to pressure P in the I frame for bradyons and luxons as

$$P' = \gamma^2 P. \quad (26)$$

Similarly, for tachyons, the pressure transformation is

$$P' = \zeta^2 P. \quad (27)$$

We also see that pressure is not Lorentz invariant. The result for bradyons and luxons coincide with the same one found previously in Refs. [33,34]. This transformation for pressure goes in contradiction with some previous works [1–3,8,29]. The tachyon pressure transformation is derived for first time. Our thermodynamic and statistical definition for pressure transformations (26) and (27) preserves the properties of ideal gases. Thus, the temperature and pressure transformation are necessary to get an ideal gas of any of these particles in both frames. After taking into account Eq. (24) and Eq. (26), we can write Eq. (23) in the I' frame as

$$P' V' = N T'. \quad (28)$$

From this we can conclude that in every inertial frame an ideal gas behaves like an ideal gas under Lorentz transformations as one would expect due to special relativity principle.

In the same token, the transformations of Eq. (24), Eq. (25), Eq. (26) and Eq. (27) for intensive quantities T and p , for bradyons, luxons or tachyons satisfy

$$\left(\frac{\partial S'}{\partial E'} \right)_{V'} dE' = \left(\frac{\partial S}{\partial E} \right)_V dE, \quad (29)$$

and

$$\left(\frac{\partial S'}{\partial V'} \right)_{E'} dV' = \left(\frac{\partial S}{\partial V} \right)_E dV. \quad (30)$$

Therefore, the variation of the entropy is the same in both frames, which means

$$dS = dS', \quad (31)$$

for any of the three species, which is according to all previous works.

Finally, it is easy to get from Eq. (21) the correct energy for an ideal gas of luxons as $E = 3NT$. From Eq. (20) we can obtain the correct non-relativistic energy for an ideal gas of bradyons when $m \gg T$

$$E \simeq \frac{3}{2}NT + Nm. \quad (32)$$

For an ideal gas of tachyons, it is possible to obtain the energy in the very-high temperature limit when $m \ll T$. From Eq. (22) we obtain

$$E \simeq \frac{Nm^2}{3T}. \quad (33)$$

From (33) we can see that the energy become null when the tachyon velocity and temperature goes to infinite as expected [24].

Conclusions. – We have shown a new path to get some known results of temperature transformation for a gas of non-interacting particles. Our treatment is from statistical first principles, only assuming the known space-time and energy-momentum Lorentz transformation along with Liouville theorem and Gibbs' postulate.

The temperature transformation for a classical ideal gas composed of particles which moves slower, equal or faster than light was shown explicitly. These transformations are the correct ones, at least for microcanonical ensemble, because they were derived using only the known statistical properties for each particle. In addition, Liouville theorem allow us to work in the microcanonical ensemble for any inertial frame, so the transformations obtained preserve the form of the first and second laws of thermodynamics in all inertial frames. This is a difference with the usual relativistic thermodynamics treatment, where the forms of the first and second laws are chosen in a more arbitrary way.

An interesting consequence of the transformations that we found for temperature and pressure is that the equation of state of an ideal gas is a Lorentz invariant. This is in agreement with the first postulate of special relativity as one would expect.

For tachyons, Eq. (33) is correct in the high temperature limit, when their velocity goes to infinite and their energy $E \rightarrow 0$. However, when the relative speed between frames w goes to infinite, the temperature T' goes to zero from Eq. (25). This is due to a tachyon in I frame is a bradyon in a I' frame which moves with speed greater than 1 relative to the I frame [30]. The behavior of the tachyon temperature transformation (25) is equivalent to temperature transformation of bradyons when $w \rightarrow 1$. This shows the duality bradyon-tachyon between frames moving at relative speeds greater than light.

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